SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1940. [1994: 108; 1995: 107; 1995: 205; 1996: 321] *Proposed by Ji Chen, Ningbo University, China.*

Show that if $x, y, z > 0$,

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}\,.$$

Solution by Marcin E. Kuczma, Warszawa, Poland.

Let F be the expression on the left side of the proposed inequality. Assume without loss of generality $x \geq y \geq z \geq 0$, with $y > 0$ (not excluding $z = 0$), and define:

$$\begin{aligned}
A &= (2x + 2y - z)(x - z)(y - z) + z(x + y)^2, \\
B &= (1/4)z(x + y - 2z)(11x + 11y + 2z), \\
C &= (x + y)(x + z)(y + z), \\
D &= (x + y + z)(x + y - 2z) + x(y - z) + y(x - z) + (x - y)^2, \\
E &= (1/4)(x + y)z(x + y + 2z)^2(x + y - 2z)^2.
\end{aligned}$$

It can be verified that

$$C^2(4F - 9) = (x - y)^2\big((x + y)(A + B + C) + (x + z)(y + z)D/2\big) + E.$$

This proves the inequality and shows that it becomes an equality only for $x = y = z$ and for $x = y > 0$, $z = 0$.

Comment.

The problem is memorable for me! It was my "solution" [1995: 107] that appeared first. According to someone's polite opinion it was elegant, but according to the impolite truth, it was wrong. I noticed the fatal error when it was too late to do anything; the issue was in print already.

In [1995: 205] a (correct) solution by Kee-Wai Lau appeared. Meanwhile I found two other proofs, hopefully correct, and sent them to the editor. Like Kee-Wai Lau's, they required the use of calculus and were lacking "lightness", so to say, so the editor asked [1995:206] for a "nice" solution. I became rather sceptical about the possibility of proving the result by those techniques usually considered as "nice", such as convexity/majorization arguments — just because the inequality turns into equality not only for $x = y = z$, but also for certain boundary configurations.

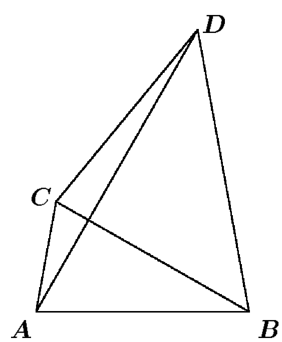
In response to the editor's prompt, Vedula Murty [1996: 321] proposed a short proof avoiding hard calculations. But I must frankly confess that I do not understand its final argument: I do not see why the sum of the first two terms in [1996: 321(3)] must be non-negative. While trying to clarify that, I arrived at the proof which I present here.

This proof can be called anything but "nice"! Decomposition into sums and products of several expressions, obviously nonnegative, and equally ugly, has the advantage that it provides a proof immediately understood and verified if one uses some symbolic calculation software (with some effort, the formula can be checked even by hand). But the striking disadvantage of such formulas is that they carefully hide from the reader all the ideas that must have led to them; they take the "background mathematics" of the reasoning away. In the case at hand I only wish to say that the equality I propose here has been inspired by Murty's brilliant idea to isolate the polynomial that appears as the third term in [1996: 321(3)] and to deal with the expression that remains.

I once overheard a mathematician problemist claiming lack of sympathy to inequality problems. In the ultimate end, he said, they all reduce to the only one fundamental inequality, which is $x^2 \geq 0$!

2124. [1996: 77] *Proposed by Catherine Shevlin, Wallsend, England.*

Suppose that $ABCD$ is a quadrilateral with $\angle CDB = \angle CBD = 50°$ and $\angle CAB = \angle ABD = \angle BCD$. Prove that $AD \perp BC$.



I. *Solution by Florian Herzig, student, Perchtoldsdorf, Austria.* (*Essentially identical solutions were submitted by Jordi Dou, Barcelona, Spain and Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany. The solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA was very similar.*)

Let F_1 and F_2 be the feet of the perpendiculars from D and A to BC respectively. Let $p = BC = CD$ and $q = AC$. Then, applying the Sine Rule to $\triangle ABC$, we have

$$CF_1 = p\cos 80°, \qquad CF_2 = q\cos 70° = \frac{p\sin 30°}{\sin 80°} = \frac{p\cos 70°}{2\sin 80°}.$$

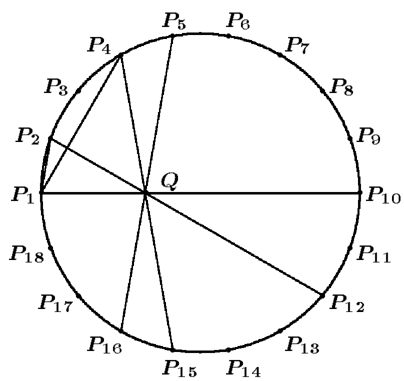
Thus we have

$$\frac{CF_1}{CF_2} = \frac{\cos 80^\circ}{\frac{\cos 70^\circ}{2\sin 80^\circ}} = \frac{2\sin 80^\circ \cos 80^\circ}{\cos 70^\circ} = \frac{\sin 160^\circ}{\sin 20^\circ} = 1.$$

Thus, $F_1 = F_2$, and this point is the intersection of AD and BC, whence $AD \perp BC$.

II. *Solution by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.*

Consider a regular 18–gon $P_1P_2\dots P_{18}$.



We will first show that P_1P_{10}, P_2P_{12} and P_4P_{15} concur.

By symmetry, P_1P_{10}, P_4P_{15} and P_5P_{16} are concurrent. Thus it is sufficient to prove that P_1P_{10}, P_2P_{12} and P_5P_{16} are concurrent.

Using the angles version of Ceva's theorem in triangle $\triangle P_1P_5P_{12}$, it if sufficient to prove that

$$\frac{\sin(\angle P_1P_{12}P_2)}{\sin(\angle P_2P_{12}P_5)} \cdot \frac{\sin(\angle P_{12}P_5P_{16})}{\sin(\angle P_{16}P_5P_1)} \cdot \frac{\sin(\angle P_5P_1P_{10})}{\sin(\angle P_{10}P_1P_{12})} = 1,$$

or

$$\frac{\sin(10^\circ)}{\sin(30^\circ)} \cdot \frac{\sin(40^\circ)}{\sin(30^\circ)} \cdot \frac{\sin(50^\circ)}{\sin(20^\circ)} = 1.$$

But this is true since

$$\begin{aligned} \sin 10^\circ \sin 40^\circ \sin 50^\circ &= \sin 10^\circ \sin 40^\circ \cos 40^\circ \\ &= \sin 10^\circ \left(\frac{\sin 80^\circ}{2}\right) = \frac{\sin 10^\circ \cos 10^\circ}{2} \\ &= \frac{\sin 20^\circ}{4} = (\sin 30^\circ)^2 \sin 20^\circ. \end{aligned}$$

So, P_1P_{10}, P_2P_{12} and P_4P_{15} concur at, say, Q.

Using this, it is easy to check that

$$\angle P_2P_4Q = \angle P_4QP_2 = 50^\circ,$$

and

$$\angle P_2P_1Q = \angle P_4QP_1 = \angle QP_2P_4 \ (= 80^\circ).$$

This information clearly determines the quadrilateral $P_1P_2P_4Q$ up to similarity, so $P_1P_2P_4Q \sim ACDB$.

Since $P_1P_4 \perp P_2Q$, it follows that $AD \perp BC$.

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SAM BAETHGE, Science Academy, Austin, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; MELETIS VASILIOU, Elefsis, Greece (two solutions); and the proposer.

The proposer writes: The genesis of this problem lies in a question asked by Junji Inaba, student, William Hulme's Grammar School, Manchester, England, in ***Mathematical Spectrum***, vol. 28 (1995/6), p. 18. He gives the diagram in my question, with the given information:

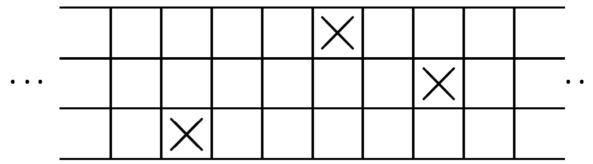
$$\begin{array}{rclcrcl} \angle CDA &=& 20^\circ, & \quad & \angle DAB &=& 60^\circ, \\ \angle DBC &=& 50^\circ, & \quad & \angle CBA &=& 30^\circ, \end{array}$$

and asks the question: "can any reader find $\angle CDB$ without trigonometry?" In fact, such a solution was given in the next issue of ***Mathematics Spectrum*** by Brian Stonebridge, Department of Computer Science, University of Bristol, Bristol, England.

The genesis of the diagram is much older, if one produces BD and AC to meet at E. See ***Mathematical Spectrum***, vol. 27 (1994/5), pp. 7 and 65–66. In one reference, the question of finding $\angle CDA$ is called "Mahatma's Puzzle", but no reference was available. Can any reader enlighten me on the origin of this puzzle?

2125. [1996: 122] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

At Lake West Collegiate, the lockers are in a long rectangular array, with three rows of N lockers each. The lockers in the top row are numbered 1 to N, the middle row $N+1$ to $2N$, and the bottom row $2N+1$ to $3N$, all from left to right. Ann, Beth, and Carol are three friends whose lockers are located as follows:



By the way, the three girls are not only friends, but also next-door neighbours, with Ann's, Beth's, and Carol's houses next to each other (in that order) on the same street. So the girls are intrigued when they notice that Beth's house number divides into all three of their locker numbers. What is Beth's house number?

Solution by Han Ping Davin Chor, student, Cambridge, MA, USA.

From the diagram, it can be observed that the lockers have numbers

$$x+3, \qquad N+x+5 \qquad \text{and} \qquad 2N+x,$$

where $1 \leq x \leq N$, x a positive integer. Here locker $x+3$ is in the first row, locker $N+x+5$ is in the second row, and locker $2N+x$ is in the third row. Let y be Beth's house number, where y is a positive integer. Since y divides into $x+3$, $N+x+5$ and $2N+x$, y must divide into

$$2(N+x+5)-(2N+x)-(x+3)=7.$$

Therefore $y = 1$ or 7. However, Beth's house is in between Ann's and Carol's. Assuming that 0 is not assigned as a house number, it means that Beth's house number cannot be 1 (else either Ann or Carol would have a house number of 0). Therefore Beth's house number is 7.

Also solved by SAM BAETHGE, Science Academy, Austin, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; J. K. FLOYD, Newnan, Georgia, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Manila, the Philippines, and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Firenze, Italy; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, New York, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOHN GRANT MCLOUGHLIN, Okanagan University College, Kelowna, B. C.; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID STONE, Georgia Southern University, Statesboro, Georgia, USA; EDWARD

T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

Two solvers eliminated 1 *as a possible answer, because the problem said that the girls were "intrigued" that Beth's house number divided all their locker numbers, which would hardly be likely if Beth's house number were just* 1*! Thus they didn't need the information about the location of Beth's house at all. Another solver, to whom the editor has therefore given the benefit of the doubt, merely stated that "the location of Ann's and Carol's houses doesn't enter into the problem".*

2126. [1996: 123] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

At Lake West Collegiate, the lockers are in a long rectangular array, with three rows of N lockers each, where N is some positive integer between 400 and 450. The lockers in the top row were originally numbered 1 to N, the middle row $N + 1$ to $2N$, and the bottom row $2N + 1$ to $3N$, all from left to right. However, one evening the school administration changed around the locker numbers so that the first column on the left is now numbered 1 to 3, the next column 4 to 6, and so forth, all from top to bottom. Three friends, whose lockers are located one in each row, come in the next morning to discover that each of them now has the locker number that used to belong to one of the others! What are (were) their locker numbers, assuming that all are three-digit numbers?

Solution by Ian June L. Garces, Ateneo de Manila University, Manila, the Philippines, and Giovanni Mazzarello, Ferrovie dello Stato, Firenze, Italy.

The friends' locker numbers are **246**, **736** and **932**.

To show this, first consider any particular locker. Then the original (before the change) number of this locker can be written as $iN + j$, where $0 \leq i \leq 2$ (the row) and $1 \leq j \leq N$ (the column). With respect to this original locker number, this particular locker has a new (after the change) number $3(j-1) + (i+1) = 3j + i - 2$.

Consider now the three friends' lockers. Since the three lockers are located one in each row, we can let them be j_1, $N + j_2$ and $2N + j_3$ where $1 \leq j_1, j_2, j_3 \leq N$. For each of these lockers, the corresponding new locker numbers will be $3j_1 - 2$, $3j_2 - 1$ and $3j_3$. Then there will be two possibilities for how their original locker numbers and their new locker numbers were "properly" interchanged:

Possibility 1. The first possibility is when

$$j_1 = 3j_3, \tag{1}$$

$$N + j_2 = 3j_1 - 2, \tag{2}$$

$$2N + j_3 = 3j_2 - 1. \tag{3}$$

Substituting (1) into (2) and solving for j_2, we have $j_2 = 9j_3 - 2 - N$. Substituting this last equality into (3) and solving for j_3, we have

$$j_3 = \frac{5N+7}{26}$$

which implies that $N \equiv 9 \bmod 26$. Choosing N between 400 and 450, we have the unique $N = 425$ and thus $j_3 = 82$, $j_2 = 311$ and $j_1 = 246$. Hence the original locker numbers are 246, 736 and 932 which, after the change, will respectively be 736, 932 and 246 which satisfy what we want.

Possibility 2. The other possibility is when

$$j_1 = 3j_2 - 1, \qquad N + j_2 = 3j_3, \qquad 2N + j_3 = 3j_1 - 2.$$

Similar computation as in Possibility 1 yields $N = 425$, $j_2 = 115$, $j_3 = 180$ and $j_1 = 344$. But this means that one of the lockers will have number 1030 which is contrary to the assumption.

Therefore, the only possible locker numbers of the three friends are 246, 736 and 932.

Also solved by SAM BAETHGE, Science Academy, Austin, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOSEPH CALLAGHAN, student, University of Waterloo, Waterloo, Ontario; HAN PING DAVIN CHOR, student, Cambridge, MA, USA; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, New York, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

Many solvers mentioned that the other set of locker numbers arising from the problem is 344, 540 *and* 1030. *Some remarked that the value of* N *was* 425 *in both cases. However, apparently nobody noticed that these two triples of numbers enjoy a curious relationship:*

$$246 + 1030 \quad = \quad 736 + 540 \quad = \quad 932 + 344\ !$$

So now readers are challenged to figure out why this relationship is true.

When $N = 425$, *the problem says that the numbers* $246, 736, 932$ *are interchanged when the lockers are renumbered. So let's call this set of numbers a "swapset" for* $N = 425$; *that is, for a particular* N, *a swapset is* *any set of numbers which get swapped among each other by the renumbering. We want true swapping; so we don't allow the sets* $\{1\}$ *or* $\{3N\}$ *(or the "middle" locker* $\{(3N+1)/2\}$ *when* N *is odd), which are obviously unchanged by the renumbering, to be in swapsets. Lots of problems concerning swapsets could be looked at. For example, one of the solvers (Stone) points out that there are no swapsets of two numbers when* $N = 425$*, but there are when* $N = 427$*: lockers* 161 *and* 481 *get swapped. Which values of* N *have swapsets of size two? Here's another problem. It's clear that the set of all numbers from* 1 *to* $3N$*, minus the two or three numbers that stay the same, will be a swapset for every* N*. But are there any numbers* N *which have* **no other** *swapsets? If so, can you describe all such* N*?*

2127. [1996: 123] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an acute triangle with circumcentre O, and D is a point on the minor arc AC of the circumcircle ($D \neq A, C$). Let P be a point on the side AB such that $\angle ADP = \angle OBC$, and let Q be a point on the side BC such that $\angle CDQ = \angle OBA$. Prove that $\angle DPQ = \angle DOC$ and $\angle DQP = \angle DOA$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

First I prove that B is an excentre of $\triangle PDQ$.

$$\begin{aligned}\angle ABC &= 180^\circ - \angle ADC\\ &= 180^\circ - (\angle ADP + \angle CDQ + \angle PDQ)\\ &= 180^\circ - (\angle CBO + \angle ABO + \angle PDQ)\\ &= 180^\circ - \angle ABC - \angle PDQ, \qquad (1)\\ \Rightarrow \angle ABC &= 90^\circ - \frac{\angle PDQ}{2},\end{aligned}$$

$$\angle PDB = \angle ADB - \angle ADP = \angle ACB - \angle OCB = \angle ACO,$$

and

$$\angle QDB = \angle CDB - \angle CDQ = \angle CAB - \angle OAB = \angle CAO.$$

Since $\triangle OAC$ is isosceles, we have that $\angle PDB = \angle QDB$ and thus BD is the internal angle bisector of $\angle PDQ$. (2)

What is more, we know that, in any $\triangle XYZ$, the excentre, M, (whose excircle touches YZ), is exactly the point on the internal angle bisector of $\angle YXZ$ outside the triangle for which

$$\begin{aligned}\angle YMZ &= 180^\circ - \angle MZY - \angle MYZ\\ &= \frac{\angle Y}{2} + \frac{\angle Z}{2} = \frac{180^\circ - \angle X}{2} = 90^\circ - \frac{\angle X}{2}.\end{aligned}$$

Therefore B is an excentre of $\triangle PDQ$ because of (1) and (2). Then BP and BQ are the external angle bisectors of $\angle DPQ$ and $\angle DQP$, respectively, whence

$$\angle APD = \angle BPQ \qquad \text{and} \qquad \angle CQD = \angle BQP. \qquad (3)$$

Starting with

$$\angle BOC = 2\angle BDC$$

we obtain

$$\begin{aligned}
180^\circ - 2\angle OBC &= 2\angle BDC, \\
90^\circ - \angle OBC &= \angle BDC, \\
180^\circ - \angle BDC &= 90^\circ + \angle OBC, \\
\angle BCD + \angle DBC &= 90^\circ + \angle ADP, \\
(180^\circ - \angle DAP) + \angle DBC &= 90^\circ + (180^\circ - \angle DAP - \angle APD), \\
\angle DBC &= 90^\circ - \angle APD, \\
\angle DOC &= 180^\circ - (\angle APD + \angle BPQ) \\
&\qquad\qquad\qquad [\text{ because of (3) }] \\
&= \angle DPQ,
\end{aligned}$$

and analogously $\angle DOA = \angle DQP$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HAN PING DAVIN CHOR, student, Cambridge, MA, USA; P. PENNING, Delft, the Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2128. [1996: 123] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a square. Let P and Q be interior points on the sides BC and CD respectively, and let E and F be the intersections of PQ with AB and AD respectively. Prove that

$$\pi \le \angle PAQ + \angle ECF < \frac{5\pi}{4}.$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

In cartesian coordinates, let $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, $D = (0,1)$, $P = (1,p)$ and $Q = (q,1)$, where $0 < p, q < 1$.

Then $E = \left(\frac{1-pq}{1-p}, 0\right)$ and $F = \left(0, \frac{1-pq}{1-q}\right)$, $\tan\angle PAB = p$, $\tan\angle DAQ = q$, $\tan\angle DCF = FD = \frac{q(1-p)}{1-q}$, $\tan\angle BCE = BE = \frac{p(1-q)}{1-p}$.

Since

$$\angle PAQ = \frac{\pi}{2} - \angle PAB - \angle DAQ \text{ and } \angle ECF = \frac{\pi}{2} + \angle DCF + \angle BCE,$$

it follows that

$$\begin{aligned}
&\angle PAQ + \angle ECF \\
&\quad = \pi + \arctan\frac{q(1-p)}{1-q} - \arctan q + \arctan\frac{p(1-q)}{1-p} - \arctan p \\
&\quad = \pi + \arctan\left(\frac{(1-pq)(p-q)^2}{(1-p)(1-q)(1-pq)^2 + (p(1-q)^2 + q(1-p)^2)(p+q)}\right)
\end{aligned}$$

by the addition formula for arctangents. Since $0 < p, q < 1$, it suffices to show that

$$0 \leq (1-pq)(p-q)^2 < (p(1-q)^2 + q(1-p)^2)(p+q).$$

The left inequality is obviously true, while the right follows from the identity

$$(p(1-q)^2 + q(1-p)^2)(p+q) = (1-pq)(p-q)^2 + 2pq((1-p)^2 + (1-q)^2).$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOSEPH CALLAGHAN, student, University of Waterloo; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VICTOR OXMAN, University of Haifa, Haifa, Israel; and the proposer.

2130. [1996: 123] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

A and B are fixed points, and ℓ is a fixed line passing through A. C is a variable point on ℓ, staying on one side of A. The incircle of ΔABC touches BC at D and AC at E. Show that line DE passes through a fixed point.

Solution by Mitko Kunchev, Baba Tonka School of Mathematics, Rousse, Bulgaria.

We choose the point P on ℓ with $AP = AB$. Let C be an arbitrary point of ℓ, different from P but on the same side of A. The incircle of $\triangle ABC$ touches the sides BC, AC, AB in the points D, E, F respectively. Let ED meet PB in the point Q. According to Menelaus' Theorem applied to $\triangle CBP$ and the collinear points E, D, Q, we get

$$\frac{PE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BQ}{QP} = 1. \tag{1}$$

We have $EC = CD$ (because they are tangents from C). Similarly, $AF = AE$, so that $FB = EP$ (since $AB = AP$). But also, $FB = DB$, so that $DB = PE$. Setting $EC = CD$ and $DB = PE$ in (1), we conclude that $BQ = QP$; therefore Q is the mid-point of BP. Hence the line DE passes through the fixed point Q.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA (two solutions); and the proposer.

Seimiya and Yiu used the same argument as Kunchev. Seimiya mentions that the result is easily shown to hold also when C coincides with P (even though the featured argument breaks down). Yiu extends the result to include excircles: The line joining the points where an excircle touches the segment BC and the line ℓ also passes through Q.

2131. [1996: 124] *Proposed by Hoe Teck Wee, Singapore.*

Find all positive integers $n > 1$ such that there exists a cyclic permutation of $(1, 1, 2, 2, \ldots, n, n)$ satisfying:

(i) no two adjacent terms of the permutation (including the last and first term) are equal; and

(ii) no block of n consecutive terms consists of n distinct integers.

Solution by the proposer.

It is clear that 2 does not have the desired property.

Suppose 3 has the specified property. So there exists a permutation of $(1, 1, 2, 2, 3, 3)$ satisfying the two conditions. WLOG assume that the first term is 1. From (ii) we know that the second term is not 1, say it is 2. From (i) the third term must be 1. From (i) and (ii) the fourth term must be 2. This leaves the two 3s as the last two terms, contradicting (i).

Suppose 4 has the specified property. So there exists a permutation of $(1, 1, 2, 2, 3, 3, 4, 4)$ satisfying the two conditions. Arrange these eight (permuted) numbers in a circle in that order so that they are equally spaced. Then the two conditions still hold. Now consider any four consecutive numbers on the circle. If they consist of only two distinct integers, we may assume by (i) that WLOG these four numbers are $1, 2, 1, 2$ in that order, and that the other four numbers are $3, 4, 3, 4$. Then (ii) does not hold. If they consist of three distinct integers, by (i) and (ii) we may assume WLOG that these four numbers are (a) $1, 2, 3, 1$ or (b) $1, 2, 1, 3$ or (c) $1, 2, 3, 2$, in these orders. By reversing the order, (c) reduces to (b). Next consider (a). If the next number is 2, then by (ii) we have $1, 2, 3, 1, 2, 3$, and the two 4s are adjacent, contradicting (i). If the next number is 3, reversing the order to obtain $3, 1, 3, 2, 1$ reduces it to (b). Finally consider (b). By (i) and (ii) the next number must be 2, followed by 3, so the two 4s are adjacent, contradicting (ii).

Next consider the following permutation for $n > 4$:

$$(4, 5, \ldots, n, 1, 2, 3, 2, 3, 4, 5, \ldots, n).$$

Clearly, (i) is satisfied. (ii) follows from the fact that there does not exist a set of four consecutive terms which is a permutation of $(1, 2, 3, 4)$.

In conclusion, the answer is: $n > 4$.

Also solved by HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA. There was one incomplete solution.

2132. [1996: 124] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Let n be an even number and z a complex number.

Prove that the polynomial $P(z) = (z+1)^n - z^n - n$ is not divisible by $z^2 + z + n$.

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California, USA.*

Let $Q(z) = z^2 + z + n$. For $n = 0$ or 1, we have that $P(z) = 0$, which is clearly divisible by $Q(z)$. For *any* $n > 1$, suppose that $P(z)$ is divisible by $Q(z)$. Then $Q(n)$ divides $P(n)$.

But $Q(n) = n(n+2) \equiv 0 \pmod{n}$, while $P(n) = (n+1)^n - n^2 - n \equiv 1 \pmod{n}$. Thus $P(z)$ is not divisible by $Q(z)$.

II. *Composite solution by F.J. Flanigan, San Jose State University, San Jose, California, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $D(z) = z^2 + z + n$. If $n = 0,\ 1$, then $P(z) = 0$, which is divisible by $D(z)$. If $n = 2$, then $P(z) = 2z - 1$, which is clearly not divisible by $z^2 + z + 2$.

For $n > 2$, suppose that $D(z)$ divides $P(z)$. Then, since $D(z)$ is monic, $P(z) = Q(z)D(z)$, where $Q(z)$ is a polynomial of degree $n - 3$ with integer coefficients. Thus $P(0) = Q(0)D(0)$, or $1 - n = nQ(0)$, which is clearly impossible.

III. *Solution and generalization by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let $n \geq 2$ be an even integer. We shall prove that if a, b, c, are complex numbers such that $a \neq 0$, then the polynomial

$$P(z) = (z+b)^n - z^n - a$$

is not divisible by $z^2 + bz + c$.

The proposer's result, which does not hold for $n = 0$, is obtained when $a = c = n$ and $b = 1$.

Let z_1 and z_2 denote the (not necessarily distinct) roots of $z^2 + bz + c$. The $z_1 + z_2 = -b$, so that $P(z_1) = z_2^n - z_1^n - a$, and $P(z_2) = z_1^n - z_2^n - a$. Since $P(z_1) + P(z_2) = -2a \neq 0$, our result follows.

The example $(z+1)^6 - z^6 = (z^2 + z + 1)(6z^3 + 9z^2 + 5z + 1)$ shows that the condition $a \neq 0$ cannot be dropped.

Also solved by: CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; NORVALD MIDTTUN (two solutions), Royal Norwegian Naval Academy, Norway; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

Besides the solvers listed in Solutions I *and* II *above, only Janous observed and showed that the assertion holds for all* $n \geq 2$.

2134*. [1996: 124] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let $\{x_n\}$ be an increasing sequence of positive integers such that the sequence $\{x_{n+1} - x_n\}$ is bounded. Prove or disprove that, for each integer $m \geq 3$, there exist positive integers $k_1 < k_2 < \ldots < k_m$, such that $x_{k_1}, x_{k_2}, \ldots, x_{k_m}$ are in arithmetic progression.

Solution by David R. Stone, Georgia Southern University, Statesboro, Georgia, USA, and Carl Pomerance, University of Georgia, Athens, Georgia, USA.

An old and well-known result of van der Waerden [4] is that if the natural numbers are partitioned into two subsets, then one of the subsets has arbitrarily long arithmetic progressions. It is not very difficult to show [1] that van der Waerden's theorem has the following equivalent formulation:

> for every number B and positive integer m, there is a number $W(m, B)$ such that if $n \geq W(m, B)$ and $0 < a_1 < a_2 < \ldots < a_n$ are integers with each $a_{i+1} - a_i \leq B$, then m of the a_i's form an arithmetic progression.

Thus, for the problem as stated, if we let B be the bound on the differences $x_{n+1} - x_n$, then for any given $m \geq 3$, there exists a $W(m, B)$ with the property stated above. Then, for any $n \geq W(m, B)$, any finite subsequence of length n will have an arithmetic progression of length m as a sub-subsequence. That is, the original sequence contains infinitely many arithmetic progressions of length m.

In 1975, Szémeredi [3] proved a conjecture of Erdős and Turán which improves on van der Waerden's Theorem, relaxing the condition that the sequence's differences have a uniform upper bound, requiring only that the sequence have a positive upper density. Hence the problem posed here also follows from the theorem of Szémeredi, who, we believe, received (for this result) the highest cash prize ever awarded by Pál Erdős — \$1,000.

Comment by the solvers.

Do we know how Pompe became interested in this problem?

References

[1] T.C. Brown, Variations on van der Waerden's and Ramsey's theorems, *Amer. Math. Monthly* **82** (1975), 993–995.

[2] Carl Pomerance, Collinear subsets of lattice point sequences — an analog of Szemeredi's Theorem, *J. Combin. Theory* **28** (1980), 140–149.

[3] E. Szemeredi, On sets of integers containing no k elements in arithmetic progression, *Acta Arith.* **27** (1975), 199–245.

[4] B.L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw. Arch. Wisk.* **15** (1928), 212–216.

Also solved by THOMAS LEONG, Staten Island, NY, USA; and JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; both using van der Waerden's theorem or its variation. Leong gave the reference: Ramsey Theory *by R.L. Graham, B.L. Rothschild and J.H. Spencer. Schlosberg remarked that van der Waerden's theorem was discussed in the July 1990 issue of* Scientific American.

The proposer showed that van der Waerden's theorem follows easily from the statement of his problem. His intention (and hope) was to find a proof independent of van der Waerden's theorem. This would establish a new "proof" of the latter. In view of his comment and the solution above, it should be obvious that the two statements are equivalent, and hence such a proof is unlikely.

2135. [1996: 124] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer. Find the value of the sum

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{(k+1)!(n-k)!(n-2k)!}.$$

Solution by Florian Herzig, student, Perchtoldsdorf, Austria. [Modified slightly by the editor.]

Let S_n denote the given summation. Note that S_1 is an "empty" sum, which we shall define to be zero. We prove that $S_n = -\binom{2n}{n+2}$.

Since $\binom{2}{3} = 0$, this is true for $n = 1$. Assume that $n \geq 2$. Following standard convention, for $k = 0, 1, 2, \ldots$, let $[x^k](f(x))$ denote the coefficient of x^k in the series expansion of the function $f(x)$. Let

$$P(x) \;=\; \left(1-x^2\right)^{n+1}(1-x)^{-(n+1)}.$$

Then, by the binomial expansion, its generalization (by Newton), and the well-known fact that $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$, we have:

$$\begin{aligned}\left[x^{2k+2}\right]\left(\left(1-x^2\right)^{n+1}\right) &= \left[\left(x^2\right)^{k+1}\right]\left(\left(1-x^2\right)^{n+1}\right)\\ &= (-1)^{k+1}\binom{n+1}{k+1},\end{aligned}$$

for $k = -1, 0, 1, 2, \ldots$, and

$$\begin{aligned}\left[x^{n-2k}\right]\left((1-x)^{-(n+1)}\right) &= (-1)^{n-2k}\binom{-n-1}{n-2k}\\ &= \binom{2n-2k}{n-2k}\end{aligned}$$

for $k \le n/2$. Hence

$$\begin{aligned}\left[x^{n+2}\right](P(x)) &= \left[x^{2k+2}\cdot x^{n-2k}\right](P(x))\\ &= \sum_{k=-1}^{\lfloor n/2\rfloor}(-1)^{k+1}\binom{n+1}{k+1}\binom{2n-2k}{n-2k}.\end{aligned}$$

On the other hand, since $P(x) = (1+x)^{n+1}$, we have $\left[x^{n+2}\right](P(x)) = 0$. Therefore

$$\sum_{k=-1}^{\lfloor n/2\rfloor}(-1)^{k+1}\binom{n+1}{k+1}\binom{2n-2k}{n-2k} \;=\; 0.$$

Since

$$S_n \;=\; -\frac{1}{n+1}\sum_{k=1}^{\lfloor n/2\rfloor}(-1)^{k+1}\binom{n+1}{k+1}\binom{2n-2k}{n-2k},$$

we get

$$\begin{aligned}S_n &= -\frac{1}{n+1}\left\{\binom{n+1}{1}\binom{2n}{n} - \binom{n+1}{0}\binom{2n+2}{n+2}\right\}\\ &= \frac{(2n+2)!}{(n+1)(n+2)!n!} - \frac{(2n)!}{n!n!}\\ &= \frac{\{2(2n+1)-(n+2)(n+1)\}}{(n+2)!n!}\times(2n)!\\ &= \frac{-n(n-1)\,(2n)!}{(n+2)!n!} \;=\; \frac{-(2n)!}{(n+2)!(n-2)!}\\ &= -\binom{2n}{n+2}.\end{aligned}$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. The correct answer, without a proof, was sent in by RICHARD I. HESS, Rancho Palos Verdes, California, USA.

If, in the given summation, one lets k start from zero (this was, in fact, the proposer's original idea), then it is easy to see that the answer becomes

$$\frac{1}{n+2}\binom{2n+2}{n+1},$$

the $(n+1)$–th Catalan number.

2136. [1996: 124] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let a, b, c be the lengths of the sides of a triangle. Given the values of $p = \sum a$ and $q = \sum ab$, prove that $r = abc$ can be estimated with an error of at most $r/26$.

Solution by P. Penning, Delft, the Netherlands.

Scale the triangle down by a factor $(a+b+c)$. The value of p then becomes 1, the value of q becomes $Q = \dfrac{q}{(a+b+c)^2}$, and $R = \dfrac{r}{(a+b+c)^3}$.

Introduce $s = \dfrac{a+b}{2}$ and $v = \dfrac{a-b}{2}$:

$$a = s+v; \qquad b = s-v; \qquad c = 1-2s;$$

$$Q = 2s - 3s^2 - v^2; \qquad R = (1-2s)(s^2-v^2).$$

Since a, b, c, represent the sides of a triangle, we must require

$$0 < c < a+b \qquad \text{and} \qquad -c < a-b < c.$$

[Ed: in other words, the triangle is not degenerate — a case which must be discarded as inappropriate.]

This translates to

$$\frac{1}{4} < s < \frac{1}{2} \qquad \text{and} \qquad |v| < \frac{1}{2} - s.$$

Lines of constant Q are ellipses in the s–v plane, with centre $s = \frac{1}{3}$, $v = 0$. So we write:

$$s = \frac{1}{3} + A\cos(x); \quad v = \sqrt{3}A\sin(x),$$

with $A = \dfrac{\sqrt{(1-3Q)}}{3}$ replacing Q.

Very symmetrical expressions are now obtained for a, b, c:

$$a = \frac{1}{3} - 2A\cos(120^\circ + x); \quad b = \frac{1}{3} - 2A\cos(120^\circ - x); \quad c = \frac{1}{3} - 2A\cos(x).$$

$$R = abc - \frac{1}{27} - A^2 - 2A^3\cos(3x).$$

Now, R is minimal for $x = 0$:

$$R_{\text{min}} = \frac{1}{27} - A^2 - 2A^3.$$

R is maximal for $x = 60°$ provided that $a \leq \frac{1}{2}$, $A \leq \frac{1}{12}$:

$$R_{\text{max}} = \frac{1}{27} - A^2 + 2A^3.$$

For $\frac{1}{12} \leq A \leq \frac{1}{6}$, we have $\cos(120° + x_{\text{max}}) = -\frac{1}{12A}$, since the maximum of a is $\frac{1}{2}$. So

$$\cos(3x_{\text{max}}) = -\frac{4}{(12A)^3} + \frac{3}{(12A)};$$

$$R_{\text{max}} = \frac{1}{27} - A^2 - 2A^3\left(-\frac{4}{(12A)^3} + \frac{3}{(12A)}\right) = \frac{1}{24} - \frac{3A^2}{2}.$$

We must determine the reciprocal of the relative spread in R:

$$F = \frac{R_{\text{max}} + R_{\text{min}}}{R_{\text{max}} - R_{\text{min}}}.$$

For $A \leq \frac{1}{12}$, we have

$$F = \frac{\frac{1}{27} - A^2}{2A^3}.$$

The minimum in F is reached at $A = \frac{1}{12}$, so that $F_{\text{min}} = 26$.

For $\frac{1}{12} \leq A \leq \frac{1}{6}$, both R_{max} and R_{min} are zero for $A = \frac{1}{6}$. So

$$\begin{aligned} R_{\text{min}} &= \left(\frac{1}{6} - A\right)\left(\frac{2}{9} + \frac{4A}{3} + 2A^2\right), \\ R_{\text{max}} &= \left(\frac{1}{6} - A\right)\left(\frac{1+6A}{4}\right). \end{aligned}$$

The minimum in F is also at $A = \frac{1}{6}$ and yields the same value for $F_{\text{min}} = 26$.

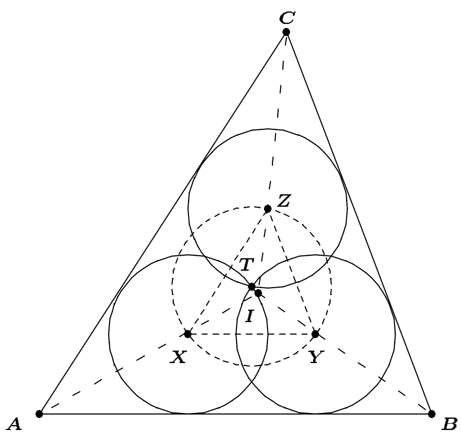
Also solved by NIELS BEJLEGAARD, Stavanger, Norway; and the proposer. One incorrect submission was received in that the sender assumed that a degenerate triangle disproved the proposition.

2137. [1996: 124, 317; 1997: 48] *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Three circles of (equal) radius t pass through a point T, and are each inside triangle ABC and tangent to two of its sides. Prove that:

(i) $t = \dfrac{rR}{R+r}$, (ii) T lies on the line segment joining the centres of the circumcircle and the incircle of $\triangle ABC$.

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.



We denote the centres of the three circles by X, Y and Z. Since the three circles pass through a common point T and have equal radius t, it follows that X, Y and Z lie on the circle with centre T and radius t. Since each of the circles is tangent to two sides of $\triangle ABC$, it follows that X, Y and Z lie on the internal bisectors of $\angle A$, $\angle B$ and $\angle C$. Since AB is a common tangent of two intersecting circles with radius t, it follows that $AB \| XY$, and analogously, we have $YZ \| BC$ and $ZX \| AC$. This implies that the lines AX, BY and CZ are bisectors of the angles of $\triangle XYZ$ as well, and so $\triangle ABC$ and $\triangle XYZ$ have the same incentre I.

Thus we conclude that triangles $\triangle ABC$ and $\triangle XYZ$ are homothetic with I as centre of similitude. This implies that:

(i) the ratio of the radii of the circumcircles of $\triangle ABC$ and $\triangle XYZ$ equals the ratio of the radii of the incircles of the triangles; that is

$$R : t \;=\; r : (r-t) \qquad Rr - Rt \;=\; rt$$
$$t(R+r) \;=\; Rr \qquad t = \frac{Rr}{R+r};$$

(ii) as corresponding points in the homothety, T (the circumcentre of $\triangle XYZ$) and the circumcentre of $\triangle ABC$ lie collinear with I, as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HAN PING DAVIN CHOR, student, Cambridge, MA, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA;

P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

Janous has seen both parts of the problem before; although unable to provide a reference to part (i), *he reconstructed the argument that he had seen, which was much like our featured solution. He, among several others, noted that* (ii) *is essentially problem 5 of the 1981 IMO* [1981: 223], *solution on pp. 35–36 of M.S. Klamkin,* International Mathematical Olympiads 1979–1985, *MAA, 1986. See also the "generalization" 694* [1982: 314] *and the related problem 1808* [1993: 299].

2138. [1996: 169] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

ABC is an acute angle triangle with circumcentre O. AO meets the circle BOC again at A', BO meets the circle COA again at B', and CO meets the circle AOB again at C'.

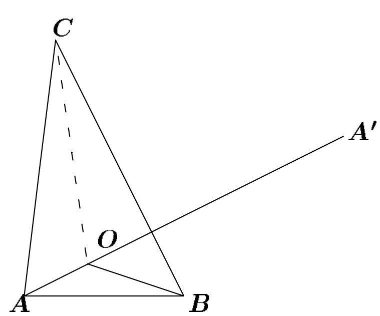
Prove that $[A'B'C'] \geq 4[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

There is an even sharper inequality:

$$[A'B'C'] \;\geq\; 3\sqrt[3]{\prod_{cyclic} \frac{\cos^2(B-C)}{\sin A \sin 2A}}\,[ABC].$$

For this, we first represent $[A'B'C']$ as a function of A, B, C and R (the circumradius).



We have: $\angle AOB = 2C$, so that $\angle BOA' = 180^\circ - 2C$ and $\angle BA'O = \angle BCO = 90^\circ - A$.

[Both angles subtend the line BO on circle BOC!] Thus,

$$\begin{aligned}\angle A'BO &= 180^\circ - (180^\circ - 2C) - (90^\circ - A)\\ &= A + 2C - 90^\circ\\ &= 180^\circ - B + C - 90^\circ\\ &= 90^\circ - (B - C).\end{aligned}$$

Hence, using the law of sines in $\triangle OBA'$, we get

$$\frac{|OA'|}{\sin(90^\circ - (B-C))} = \frac{R}{\sin(90^\circ - A)};$$

that is, $|OA'| = \dfrac{R\cos(B-C)}{\cos A}$.

Similarly, $|OB'| = \dfrac{R\cos(C-A)}{\cos B}$ and $|OC'| = \dfrac{R\cos(A-B)}{\cos C}$.

Now, since $\angle A'OB' = \angle AOB = 2C$, we get, via the trigonometric area formula of triangles, that

$$\begin{aligned}[A'B'O] &= \tfrac{1}{2}|OA'|\cdot|OB'|\cdot\sin(\angle A'OB') \\ &= \frac{R^2}{2}\frac{\cos(B-C)\cos(C-A)}{\cos A\cos B}\sin 2C,\end{aligned}$$

and similarly for $[B'C'O]$ and $[C'A'O]$. Thus

$$\begin{aligned}[A'B'C'] &= [A'B'O]+[B'C'O]+[C'A'O] \\ &= \left(\sum_{cyclic}\frac{\cos(C-A)\cos(C-B)}{\cos A\cos B}\sin 2C\right)\times\frac{R^2}{2}. \qquad (1)\end{aligned}$$

Next, we recall the formula

$$[ABC] = 2R^2\prod_{cyclic}\sin A \qquad (2)$$

From (1), we get, via the arithmetic-geometric-mean inequality:

$$\begin{aligned}&\sum_{cyclic}\frac{\cos(C-A)\cos(C-B)}{\cos A\cos B}\sin 2C \\ &\geq 3\left[\prod_{cyclic}\left(\frac{\cos^2(B-C)}{\cos^2 A}\cdot\sin 2A\right)\right]^{\frac{1}{3}} \\ &= 6\left[\prod_{cyclic}\frac{\cos^2(B-C)}{\cos A}\cdot\sin A\right]^{\frac{1}{3}} \\ &= 6\left[\prod_{cyclic}\frac{\cos^2(B-C)}{\sin^2 A\cos A}\right]^{\frac{1}{3}}\times\prod_{cyclic}\sin A \\ &= 12\left[\prod_{cyclic}\frac{\cos^2(B-C)}{\sin A\sin 2A}\right]^{\frac{1}{3}}\times\prod_{cyclic}\sin A;\end{aligned}$$

so that, using (1) and (2),

$$[A'B'C'] \geq 3\left[\prod_{cyclic}\frac{\cos^2(B-C)}{\sin A\sin 2A}\right]^{\frac{1}{3}}\times[ABC]$$

as claimed.

Finally, we recall the angle inequality:

$$\prod_{cyclic} \cos^2(B-C) \geq \frac{512}{27} \prod_{cyclic} (\sin^2 A \cos A) \left[= \frac{64}{27} \prod_{cyclic} (\sin A \sin 2A)\right]$$

which is valid for all triangles (but interesting only for acute triangles) with equality if and only if $A = B = C = 60^\circ$, or the degenerate cases with two of A, B, C being right angles. This immediately yields

$$\left[\prod_{cyclic} \frac{\cos^2(B-C)}{\sin A \sin 2A}\right]^{\frac{1}{3}} \geq \sqrt[3]{\frac{64}{27}} = \frac{4}{3},$$

and the original inequality follows.

Also solved by D.J. SMEENK, Zaltbommel, the Netherlands.

2139. [1996: 169, 219] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Point P lies inside triangle ABC. Let D, E, F be the orthogonal projections from P onto the lines BC, CA, AB, respectively. Let O' and R' denote the circumcentre and circumradius of the triangle DEF, respectively. Prove that

$$[ABC] \geq 3\sqrt{3}R'\sqrt{R'^2 - (O'P)^2},$$

where $[XYZ]$ denotes the area of triangle XYZ.

Solution by the proposer.

Let $\mathcal{C}$ denote the circumcircle of DEF. Let P' be the symmetric point to P with respect to O'. Let $\mathcal{E}$ be the ellipse with foci P and P' tangent (internally) to $\mathcal{C}$. The diameter of the ellipse $\mathcal{E}$ is $2R'$, and its area is equal to $\pi R'\sqrt{R'^2 - (O'P)^2}$. Since the locus of the orthogonal projections from P onto tangents to the ellipse $\mathcal{E}$ is the circle $\mathcal{C}$, the sides of ABC must be tangent to $\mathcal{E}$. Thus $\mathcal{E}$ is inscribed in the triangle ABC. Let L be an affine mapping which takes $\mathcal{E}$ to some circle of radius R, and let it take the triangle ABC to the triangle $A'B'C'$. Since L preserves the ratio of areas, we obtain

$$\frac{[ABC]}{\pi R'\sqrt{R'^2 - (O'P)^2}} = \frac{[ABC]}{\text{area of } \mathcal{E}} = \frac{[A'B'C']}{\pi R^2} \geq \frac{3\sqrt{3}R^2}{\pi R^2}, \qquad (1)$$

since among all triangles circumscribed about the given circle, the one of smallest area is the equilateral triangle. Thus (1) is equivalent to the desired inequality, so we are done.

Remarks: The same proof works also for an n-gon which has an interior point whose projections onto the sides of the n-gon are concyclic. The analogous inequality will be

$$[A_1 \ldots A_n] \geq n \tan\left(\frac{\pi}{n}\right) R' \sqrt{R'^2 - (O'P)^2}\,.$$

Note that as a special case, when the n-gon has the incircle and $P = O'$ we obtain the well-known result that *among all n-gons circumscribed about a given circle, the one of smallest area is the regular one*, though it is used in the proof.

2140. [1996: 169] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Determine the quartic $f(x) = x^4 + ax^3 + bx^2 + cx - c$ if it shares two distinct integral zeros with its derivative $f'(x)$ and $abc \neq 0$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let the zeros of $f(x)$ be the integers p and q; without loss of generality $p > q$. It is a well-known theorem that if a polynomial $Q(x)$ divides the polynomial $P(x)$ as well as the derivative $P'(x)$, then $(Q(x))^2$ divides $P(x)$. Applying the theorem for this problem, we obtain

$$f(x) = (x-p)^2(x-q)^2 = x^4 + ax^x + bx^2 + cx - c.$$

Comparing coefficients of x and the constant term yields

$$0 = c + (-c) = -2(p+q)pq + p^2q^2.$$

As $pq = 0$ implies $abc = 0$, we may divide by pq

$$\begin{aligned} pq - 2p - 2q &= 0 \\ (p-2)(q-2) &= 4 = 4 \cdot 1 = (-1)(-4) \end{aligned}$$

Hence $(p,q) = (6,3) \vee (1,-2)$ (since $p \neq q$) and the two possible polynomials are

$$\begin{aligned} f_1(x) &= (x-6)^2(x-3)^2 = x^4 - 18x^3 + 117x^2 - 324x + 324, \\ f_2(x) &= (x+2)^2(x-1)^2 = x^4 + 2x^3 - 3x_4^2x + 4. \end{aligned}$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State

University, Big Rapids, Michigan, USA; BEATRIZ MARGOLIS, Paris, France; L. RICE, Woburn Collegiate, Scarborough, Ontario; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer. There were eight incorrect or incomplete solutions.

Do you know the equation of this graph?

Contributed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.